EKT241 - Electromagnetic Theory

Chapter 3 - Electrostatics
Chapter Objectives

✓ Application of Maxwell’s equations
✓ Using of Coulomb’s law
✓ Using of Gauss’s law
✓ Conductivities of metals
✓ Boundary conditions that leads to tangential components of $\mathbf{D}$, $\mathbf{E}$, and $\mathbf{J}$
✓ Capacitance of a two-conductor body
✓ Electrostatic energy density stored in a dielectric medium
Chapter Outline

3-1) Maxwell’s Equations
3-2) Charge and Current Distributions
3-3) Coulomb’s Law
3-4) Gauss’s Law
3-5) Electric Scalar Potential
3-6) Electrical Properties of Materials
3-7) Conductors
3-8) Dielectrics
3-9) Electric Boundary Conditions
3-10) Capacitance
3-11) Electrostatic Potential Energy
3-12) Image Method
Maxwell Equations

- Maxwell Equations consist of four differential equations,
- (1) the law of electro-magnetic induction (Faraday),
- (2) Ampere-Maxwell's law,
- (3) Gauss's law of electric field, and
- (4) Gauss's law of magnetic field.

\[ \nabla \cdot D = \rho_v \]
\[ \nabla \times E = -\frac{\partial B}{\partial t} \]
\[ \nabla \cdot B = 0 \]
\[ \nabla \times H = H + \frac{\partial D}{\partial t} \]

(1) \[ \nabla \times E = -\frac{\partial B}{\partial t} \]
(2) \[ \nabla \times H = H + \frac{\partial D}{\partial t} \]

where \( D \) is electric flux, \( B \) is magnetic flux density, and \( i \) is electric current density which will be explained later.

The latter two equations are expressed in terms of an operation called "divergence" of \( D \) and of \( B \).
3-1 Maxwell’s Equations

- Maxwell’s equations:

  Relationship:  **Those equations describe the interrelationship between electric field, magnetic field, electric charge, and electric current.**

  \[
  \begin{align*}
  D &= \varepsilon E \\
  B &= \mu H
  \end{align*}
  \]

  where \( E, D \) = electric field quantities
  
  - \( \varepsilon \) = electrical permittivity of the material
  - \( \mu \) = magnetic permeability of the material
  
  \( \rho_v \) = electric charge density per unit volume
  
  \( J \) = current density per unit area
3-1 Maxwell’s Equations

- In **static** case, $\partial/\partial t = 0$.
- Maxwell’s equations is reduced to:

**Electrostatics**

\[
\nabla \cdot D = \rho_v \\
\nabla \times E = 0
\]

**Magnetostatics**

\[
\nabla \cdot B = 0 \\
\n\nabla \times H = J
\]
Maxwell’s Equations in Vacuum

In a vacuum there are no true charges or currents, so that \( \rho \) and \( I \) are both zero. Then Maxwell’s equations become,

- **Gauss’s Law (electrical)**
  \[
  \int_{A} \vec{E} \cdot d\vec{A} = 0
  \]

- **Gauss’s Law (magnetic)**
  \[
  \int_{A} \vec{B} \cdot d\vec{A} = 0
  \]

- **Faraday’s Law**
  \[
  \oint_{C} \vec{E} \cdot d\ell = -\frac{d\phi_{B}}{dt}
  \]

- **Ampere’s Law**
  \[
  \oint_{C} \vec{B} \cdot d\ell = \mu_{0} \varepsilon_{0} \frac{d\phi_{E}}{dt}
  \]
Gauss' Law for Electricity

- The electric flux out of any closed surface is proportional to the total charge enclosed within the surface.
- The integral form of Gauss' Law finds application in calculating electric fields around charged objects.
- In applying Gauss' law to the electric field of a point charge, one can show that it is consistent with Coulomb's law.
- While the area integral of the electric field gives a measure of the net charge enclosed, the divergence of the electric field gives a measure of the density of sources.
Gauss' Law for Magnetism

- The net \textbf{magnetic flux} out of any closed surface is zero. This amounts to a statement about the sources of magnetic field. For a magnetic dipole, any closed surface the magnetic flux directed inward toward the south pole will equal the flux outward from the north pole. The net flux will always be zero for dipole sources. If there were a magnetic monopole source, this would give a non-zero \textbf{area integral}.

\begin{align*}
\int \mathbf{B} \cdot d\mathbf{A} &= 0 \\
\nabla \cdot \mathbf{B} &= 0
\end{align*}
Faraday’s Law of Induction

- The **line integral** of the **electric field** around a closed loop is equal to the negative of the rate of change of the **magnetic flux** through the area enclosed by the loop.

- This line integral is equal to the **generated voltage** or **emf** in the loop, so Faraday's law is the basis for **electric generators**. It also forms the basis for **inductors** and **transformers**.

\[
\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}
\]

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}
\]
Ampere's Law

- In the case of static electric field, the line integral of the magnetic field around a closed loop is proportional to the electric current flowing through the loop. This is useful for the calculation of magnetic field for simple geometries.
3-2 Charge and Current Distributions

- Charge may be distributed over a volume, a surface or a line.

3-2.1 Charge Densities

- **Volume charge density** $\rho_v$ is defined as

  $$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} = \frac{dq}{dv} \quad (\text{C/m}^3)$$

- Total charge $Q$ volume $V$ is given by

  $$Q = \int V \rho_v dv \quad \text{(C)}$$
3-2.1 Charge Densities

- **Surface charge density**
  \[ \rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \quad \text{(C/m}^2\text{)} \]

- **Line charge density**
  \[ \rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \quad \text{(C/m)} \]
Example 3.1 Line Charge Distribution

Calculate the total charge $Q$ contained in a cylindrical tube of charge oriented along the $z$-axis. The line charge density is $\rho = 2z$, where $z$ is the distance in meters from the bottom end of the tube. The tube length is 10 cm.

Solution

The total charge $Q$ is

$$Q = \int_{0}^{0.1} \rho \, dz = \int_{0}^{0.1} 2z \, dz = \left[ z^2 \right]_{0}^{0.1} = 10^{-2} \text{ C}$$
3-2.2 Current Densities

- **Current density** is defined as
  \[ J = \rho_v u \quad (\text{A/m}^2) \]

- For surface \( S \), total current flowing through is
  \[ I = \int_S J \, ds \quad (\text{A}) \]

- There are 2 types of current:
  1) Convection current (generated by actual movement of electrically charged matter; does not obey Ohm’s law)
  2) Conduction current (atoms of conducting material do not move; obeys Ohm’s law)
3-3 Coulomb’s Law

- **Coulomb’s law** states that:

\[ E = \hat{R} \frac{q}{4\pi \varepsilon R^2} \quad \text{(V/m)} \]

where \( \hat{R} \) = unit vector from p to q

- For an electric field \( \mathbf{E} \) at a given point in space,

\[ F = q' \mathbf{E} \quad \text{(N)} \]

where \( \mathbf{D} = \varepsilon \mathbf{E} \)

\[ \varepsilon = \varepsilon_R \varepsilon_0 \]

\[ \varepsilon_0 = 8.85 \times 10^{-12} \approx (1/36\pi) \times 10^{-9} \text{ (F/m)} \]
3-3.1 Electric Field due to Multiple Point Charges

- Total electric field $E$ at any point in space is

$$E = \frac{1}{4\pi\varepsilon_0} \left[ \frac{q_1 (R - R_1)}{|R - R_1|^3} + \frac{q_2 (R - R_2)}{|R - R_2|^3} \right]$$

- In general for case of $N$ point of charges,

$$E = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} q_i \frac{(R - R_i)}{|R - R_i|^3} \text{ (V/m)}$$
Example 3.3 Electric Field due to Two Point Charges

Two point charges with \( q_1 = 2 \times 10^{-5} \text{C} \) and \( q_2 = -4 \times 10^{-5} \text{C} \) are located in free space at \((1, 3, -1)\) and \((-3, 1, -2)\), respectively, in a Cartesian coordinate system.

Find (a) the electric field \( \mathbf{E} \) at \((3, 1, -2)\) and (b) the force on a \( 8 \times 10^{-5} \text{C} \) charge located at that point. All distances are in meters.
The electric field $\mathbf{E}$ with $\varepsilon = \varepsilon_0$ (free space) is given by

$$E = \frac{1}{4\pi\varepsilon_0} \left[ q_1 \frac{(R - R_1)}{|R - R_1|^3} + q_2 \frac{(R - R_2)}{|R - R_2|^3} \right]$$

The vectors are $R_1 = \hat{x} + \hat{y}3 - \hat{z}$, $R_2 = -\hat{x}3 + \hat{y} - \hat{z}2$, $R = \hat{x}3 + \hat{y} - \hat{z}2$

a) Hence, $E = \frac{\hat{x} - \hat{y}4 - \hat{z}2}{108\pi\varepsilon_0} \times 10^{-5}$ (V/m)

b) We have

$$F = q_3 E = 8 \times 10^{-5} \times \frac{\hat{x} - \hat{y}4 - \hat{z}2}{108\pi\varepsilon_0} \times 10^{-5} = \frac{\hat{x}2 - \hat{y}8 - \hat{z}4}{27\pi\varepsilon_0} \times 10^{-10} \text{ (N)}$$
3-3.2 Electric Field due to Multiple Point Charges

- Total electric field due to 3 types of distribution:

\[
E = \int_v dE = \frac{1}{4\pi \varepsilon} \int_v \hat{R} \frac{\rho_v dv'}{R_{12}} \quad \text{(volume distribution)}
\]

\[
E = \int_s dE = \frac{1}{4\pi \varepsilon} \int_s \hat{R} \frac{\rho_s ds'}{R_{12}} \quad \text{(surface distribution)}
\]

\[
E = \int_l dE = \frac{1}{4\pi \varepsilon} \int_l \hat{R} \frac{\rho_l dl'}{R_{12}} \quad \text{(line distribution)}
\]
Example 3.5 Electric Field of a Circular Disk of Charge

Find the electric field at a point $P(0, 0, h)$ in free space at a height $h$ on the z-axis due to a circular disk of charge in the x–y plane with uniform charge density $\rho_s$ as shown, and then evaluate $\mathbf{E}$ for the infinite-sheet case by letting $a \rightarrow \infty$. 
solution 3.5 Electric Field of a Circular Disk of Charge

A ring of radius $r$ and width $dr$ has an area $ds = 2\pi r dr$

The charge is $dq = \rho_s ds = 2\pi \rho_s r dr$

The field due to the ring is $dE = \hat{z} \frac{h}{4\pi \varepsilon_0 (r^2 + h^2)^{3/2}} (2\pi \rho_s r dr)$

The total field at $P$ is

$$E = \hat{z} \rho_s h \int_0^a \frac{r dr}{2\varepsilon_0 (r^2 + h^2)^{3/2}} = \hat{z} \frac{\rho_s}{2\varepsilon_0} \left[ 1 - \frac{|h|}{\sqrt{a^2 + h^2}} \right]$$

For an infinite sheet of charge with $a = \infty$

$$E = \hat{z} \frac{\rho_s}{2\varepsilon_0} \text{ (infinite sheet of charge)}$$
• **Gauss’s law** states that \( \mathbf{D} \) through a surface is proportional to enclosed \( Q \).

• Differential and integral form of **Gauss’s law** is defined as

\[
\nabla \cdot \mathbf{D} = \rho_v \quad \text{(Gauss's law)}
\]

\[
\int_{S} \mathbf{D} \cdot d\mathbf{s} = Q \quad \text{(Gauss's law)}
\]
Example 3.6 Electric Field of a Line of Charge

Use Gauss’s law to obtain an expression for $\mathbf{E}$ in free space due to an infinitely long line of charge with uniform charge density $\rho_l$ along the $z$-axis.

**Solution**

Construct a cylindrical Gaussian surface and the integral is

$$\int_0^h \int_{\phi=0}^{2\pi} \hat{r} D_r \cdot \hat{r} r d\phi dz = \rho_l h \quad \text{or} \quad 2\pi h D_r r = \rho_l h$$

which gives the result

$$E = \frac{D}{\varepsilon_0} = \hat{r} \frac{D_r}{\varepsilon_0} = \hat{r} \frac{\rho_l}{2\pi \varepsilon_0 r} \quad \text{(infinite line of charge)}$$
3-5 Electric Scalar Potential

- Electric potential energy is required to move unit charge between the 2 points.

3-5.1 Electric Potential as a Function of Electric Field

- Potential difference along any path is obtained by

\[ V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} E \cdot dl \]
3-5.1 Electric Potential as a Function of Electric Field

- **Kirchhoff’s voltage law** states that net voltage drop around a closed loop is zero.
- Line integral $\mathbf{E}$ around closed contour $C$ is

\[ \oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \quad \text{(Electrostatics)} \]

- The electric potential $V$ at any point is given by

\[ V = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l} \quad (V) \]
3-5.2 Electric Potential due to Point Charges

- For $N$ discrete point charges, electric potential is

$$V(R) = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i}{|R - R_i|}$$  \hspace{1cm} (V)

3-5.3 Electric Potential due to Continuous Distributions

- For a continuous charge distribution, we have

$$V(R) = \frac{1}{4\pi\varepsilon} \int_{v'} \frac{\rho_v}{R'} dv' \text{ (volume distribution on)}$$

$$V(R) = \frac{1}{4\pi\varepsilon} \int_{s'} \frac{\rho_s}{R'} ds' \text{ (surface distribution on)}$$

$$V(R) = \frac{1}{4\pi\varepsilon} \int_{l'} \frac{\rho_l}{R'} dl' \text{ (line distribution on)}$$
3-5.4 Electric Field as a Function of Electric Potential

- To find $E$ for any charge distribution easily,

\[ E = -\nabla V \]

3-5.5 Poisson’s Equation

- Poisson’s and Laplace’s equations are used to find $V$ where boundaries are known.

\[ \nabla^2 V = -\frac{\rho_V}{\varepsilon} \quad \text{(Poisson's equation)} \]

\[ \nabla^2 V = 0 \quad \text{(Laplace's equation)} \]
Different materials have different conductivity.

<table>
<thead>
<tr>
<th>Material</th>
<th>Conductivity, $\sigma$ (S/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Conductors</strong></td>
<td></td>
</tr>
<tr>
<td>Silver</td>
<td>$6.2 \times 10^7$</td>
</tr>
<tr>
<td>Copper</td>
<td>$5.8 \times 10^7$</td>
</tr>
<tr>
<td>Gold</td>
<td>$4.1 \times 10^7$</td>
</tr>
<tr>
<td>Aluminum</td>
<td>$3.5 \times 10^7$</td>
</tr>
<tr>
<td>Iron</td>
<td>$10^7$</td>
</tr>
<tr>
<td>Mercury</td>
<td>$10^6$</td>
</tr>
<tr>
<td>Carbon</td>
<td>$3 \times 10^4$</td>
</tr>
<tr>
<td><strong>Semiconductors</strong></td>
<td></td>
</tr>
<tr>
<td>Pure germanium</td>
<td>2.2</td>
</tr>
<tr>
<td>Pure silicon</td>
<td>$4.4 \times 10^{-4}$</td>
</tr>
<tr>
<td><strong>Insulators</strong></td>
<td></td>
</tr>
<tr>
<td>Glass</td>
<td>$10^{-12}$</td>
</tr>
<tr>
<td>Paraffin</td>
<td>$10^{-15}$</td>
</tr>
<tr>
<td>Mica</td>
<td>$10^{-15}$</td>
</tr>
<tr>
<td>Fused quartz</td>
<td>$10^{-17}$</td>
</tr>
</tbody>
</table>
3-7 Conductors

- **Conductivity** of the material, $\sigma$, is defined as

$$\sigma = -\rho_{ve} \mu_e + \rho_{vh} \mu_h$$

$$= (N_e \mu_e + N_h \mu_h) e \text{ (S/m) (semiconductor)}$$

$$\sigma = -\rho_{ve} \mu_e = N_e \mu_e e \text{ (S/m) (conductor)}$$

where $\mu_e = \text{electron mobility (m/s)}$

$\mu_h = \text{hole mobility (m/s)}$

$N_e = \text{number of free electrons}$

$N_h = \text{number of free holes}$

$e = 1.6 \times 10^{-19} \text{ C (absolute charge)}$
3-7 Conductors

- **Point form of Ohm’s law** states that

\[ J = \sigma E \quad \text{(A/m}^2\text{)} \quad \text{(Ohm's law)} \]

- Properties for perfect dielectric and conductor:

  - Perfect dielectric with \( \sigma = 0 \): \( J = 0 \), regardless of \( E \)
  - Perfect conductor with \( \sigma = \infty \): \( E = 0 \), regardless of \( J \)
Example 3.8  Conduction Current in a Copper Wire

A 2-mm-diameter copper wire with conductivity of $5.8 \times 10^7$ S/m and electron mobility of 0.0032 (m²/V·s) is subjected to an electric field of 20 (mV/m).

Find (a) the volume charge density of free electrons, (b) the current density, (c) the current flowing in the wire, (d) the electron drift velocity, and (e) the volume density of free electrons.
Solution 3.8 Conduction Current in a Copper Wire

a) \[ \rho_{ve} = -\frac{\sigma}{\mu_e} = -\frac{5.8 \times 10^7}{0.0032} = -1.81 \times 10^{10} \text{ (C/m}^3\text{)} \]

b) \[ J = \sigma E = 5.8 \times 10^7 \times 20 \times 10^{-3} = 1.16 \times 10^6 \text{ (A/m}^2\text{)} \]

c) \[ I = JA = 1.16 \times 10^6 \left(\frac{\pi \times 4 \times 10^{-6}}{4}\right) = 3.64 \text{ A} \]

d) \[ u_e = -\mu_e E = -0.0032 \times 20 \times 10^{-3} = -6.4 \times 10^{-5} \text{ m/s} \]

e) \[ N_e = -\frac{\rho_{ve}}{e} = \frac{1.81 \times 10^{10}}{1.6 \times 10^{-19}} = 1.13 \times 10^{29} \text{ electrons/m}^3 \]
3-7.1 Resistance

- For resistor of arbitrary shape, resistance $R$ is

$$R = \frac{V}{I} = \frac{-\int E \cdot dl}{l} = \frac{-\int E \cdot dl}{\int J \cdot ds} = \frac{-\int E \cdot dl}{\int \sigma E \cdot ds}$$

- The reciprocal of $R$ is called the conductance $G$,

$$G = \frac{1}{R} = \frac{\sigma A}{l} \quad (S \text{ or siemens})$$
3-7.2 Joule’s Law

- Joule’s law states that for a volume $v$, the total dissipated power is

$$P = \int_E E \cdot J \, dv \quad (W) \quad (\text{Joule's law})$$

3-8 Dielectrics

- Conductor has free electrons.
- Dielectric electrons are strongly bounded.
- When $E$ exceeds critical value, the material encounter dielectric breakdown.
3-9 Electric Boundary Conditions

- Tangential component $E_t$ of the electric field is defined as

$$E_{1t} = E_{2t} \quad (\text{V/m})$$

- Boundary condition on the tangential component of $D_t$ is

$$\frac{D_{1t}}{\varepsilon_1} = \frac{D_{2t}}{\varepsilon_2}$$
3-9.1 Dielectric–Conductor Boundary

- Boundary condition at conductor surface is

\[ D_1 = \varepsilon_1 E_1 = \hat{n} \rho_s \]  (at conductor surface)

where \( \hat{n} \) = normal vector pointing outward

- Electric field lines point away surface when \( \rho_s \) is positive.
The normal component of \( \mathbf{J} \) has to be continuous across the boundary between 2 different media under electrostatic conditions.

Setting \( J_{1n} = J_{2n} \), we have

\[
J_{1n} \left( \frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2} \right) = \rho_s \quad \text{(electrostatics)}
\]
Capacitance

Consider two metallic objects connected by a voltage source.

- The total charge on both metal objects have the same magnitude but opposite sign.
- The charge is proportional to the applied voltage: $Q \propto V$.
- The constant of proportionality is called the capacitance $C$:
  $$Q = CV$$
- Capacitance has units **Coulombs/Volts** or **Farads**.
- More generally, capacitance is also defined as:
  $$C = \frac{dQ}{dV}$$


**3-10 Capacitance**

- **Capacitance** is defined as

\[ C = \frac{Q}{V} \quad (\text{C/V or F}) \]

where \( V \) = potential difference (V)
\( Q \) = charge (C)
\( C \) = capacitance (F, farads)

- For a medium with uniform \( \sigma \) and \( \varepsilon \),

\[ RC = \frac{\varepsilon}{\sigma} \]
Capacitance of Parallel Plates

From previous lectures:

\[ E_x = \frac{V}{d} \]

area = \( A \)

\[ \phi = \frac{V}{d} \]

\( \phi = 0 \)

\[ \sigma_L = \varepsilon_0 \frac{V}{d} \]

\[ \sigma_R = -\varepsilon_0 \frac{V}{d} \]

\[ Q = \frac{\varepsilon_0 A}{d} V \]

\[ C = \frac{dQ}{dV} = \frac{Q}{V} = \frac{\varepsilon_0 A}{d} \]
Example 3.11 Capacitance and Breakdown Voltage of Parallel-Plate Capacitor

Obtain an expression for the capacitance $C$ of a parallel-plate capacitor comprised of two parallel plates each of surface area $A$ and separated by a distance $d$. The capacitor is filled with a dielectric material with permittivity $\varepsilon$. Also, determine the breakdown voltage if $d = 1 \text{ cm}$ and the dielectric material is quartz.
The charge density on the upper plate is $\rho_s = Q/A$. Hence,

$$E = -\hat{z}E \quad \text{where} \quad E = \rho_s / \varepsilon = Q / \varepsilon A$$

The voltage difference is

$$V = -\int_0^d E \cdot dl = -\int_0^d (-\hat{z}E) \cdot \hat{z}dz = Ed$$

Hence, the breakdown voltage is

$$V_{br} = E_{ds}d = 30 \times 10^6 \times 10^{-2} = 3 \times 10^5 \text{ V}$$
Capacitance of Concentric Cylinders

From previous lectures:

\[ E_r(r) = -\frac{V}{r \ln\left(\frac{b}{a}\right)} \]

\[ \sigma_{out} = \varepsilon_o \frac{V}{b \ln(b/a)} \]

\[ \sigma_{in} = -\varepsilon_o \frac{V}{a \ln(b/a)} \]

Since the structure is infinite in the z-direction (i.e. in the direction coming out of the slide) one can only talk about capacitance per unit length (units: Farads/m)

\[ Q = \text{charge per unit length} = (2\pi b) \varepsilon_o \frac{V}{b \ln(b/a)} \]

\[ C = \text{capacitance per unit length} = \frac{dQ}{dV} = \frac{Q}{V} = \frac{2\pi \varepsilon_o}{\ln(b/a)} \]
3-11 Electrostatic Potential Energy

- When $q$ travels to a given point, electrostatic potential energy $W_e$ will be stored.

\[
w_e = \frac{W_e}{V} = \frac{1}{2} \varepsilon E^2 \quad (\text{J/m}^3)
\]

where $w_e = \text{electrostatic potential energy per unit volume}$

- For parallel-plate capacitor, the plates are attracted by an electrical force $F$,

\[
F = -\nabla W_e \quad (\text{N})
\]
Image theory states that a charge $Q$ above a grounded perfectly conducting plane is equal to $Q$ and its image $-Q$ with ground plane removed.
Use image theory to determine $V$ and $E$ at an arbitrary point $P(x, y, z)$ in the region $z > 0$ due to a charge $Q$ in free space at a distance $d$ above a grounded conducting plane.

**Solution**

From charge $Q(0, 0, d)$ and its image $-Q(0, 0, -d)$ in Cartesian coordinates, we have

$$E = \frac{1}{4\pi\varepsilon_0} \left( \frac{QR_1}{R_1^3} + \frac{-QR_2}{R_2^3} \right) = \frac{Q}{4\pi\varepsilon_0} \begin{bmatrix} \hat{x}x + \hat{y}y + \hat{z}(z - d) \\ \left[ x^2 + y^2 + (z - d)^2 \right]^{3/2} \\ \hat{x}x + \hat{y}y + \hat{z}(z + d) \\ \left[ x^2 + y^2 + (z + d)^2 \right]^{3/2} \end{bmatrix}$$